On Asymptotic Efficiency for Asynchronous CDMA

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***Abstract*—We consider asynchronous CDMA systems and analyze their performances in terms of asymptotic efficiency when linear receivers are employed and the system becomes overloaded. We show that in such systems, differently from synchronous systems, linear detectors can be asymptotically efficient and, specifically, that the MMSE detector is asymptotically efficient with probabilty 1. Simulations confirm that it is better to be asynchronous than synchronous and even under heavy overloading the MMSE receiver is still asymptotically efficient.**

I. Introduction

Overloaded CDMA systems are of interest because offer the possibility to have more and more users transmitting over the same band. Limitations on such systems are imposed by multiuser interference (MUI) that may determine strong degradation on each user's performance. In overloaded systems the number of users exceeds the processing gain, therefore there are not enough orthogonal signatures available for all and multiuser interference is unavoidable. Receivers need to be able to cancel out this interference that even if it can be controlled at system design level, may be very strong and make impossible to obtain vanishing error probabilities. In [1] Romano *et al.* have shown that in a synchronous overloaded CDMA system with any linear receiver, at least one user has a probability of error that does not descrese as the channel noise goes to zero, regardless of the choice of any set of signatures and amplitudes. These limitations are the consequence of the overloaded nature of the system and are not related to any other parameter. However, Luo *et al.* in [2] have pointed out that asynchronous overloaded CDMA has better performances than the corresponding synchronous system (with iterative receivers) and have proposed that for overloaded systems users should be deliberately asynchronous (at least for non fading channels). This result suggests that asynchronous systems might be better at least in some cases than corresponding synchronous systems in terms of asymptotic performances. In this paper we investigate whether limitations exists for linear receivers in overloaded asynchronous CDMA.

The paper is organized as follows. In section II the model for asynchronous CDMA is introduced, in section III the asymptotic efficiency for the MMSE detector is computed and commented. We report simulation results in section IV and we draw some conclusions in section V.

II. System Model

We consider the received signal for an asynchronous CDMA system

$$y(t) = \sum_{k=1}^{K} \sum_{i=0}^{M-1} a_k b_k(i) s_k(t - iT - \tau_k) + \sigma n(t), \quad (1)$$

where K is the number of users, M is the frame length, while for the k-th user a_k (> 0), $b_k(i) \in \{-1, +1\}$, $s_k(t)$ and τ_k represent the amplitude at the receiver, the bit transmitted at the i-th time slot, the signature and the temporal delay, respectively. Temporal delays can be though as the different effect of the channel on distinct users and here are considered as realization of same random process. Expression (1) is quite general and includes as particular cases different well-known CDMA signals depending on the nature of delays. When $\tau_k = 0$, $\forall k$, we obtain the synchronous model; with τ_k discrete random variable, that can take values equal to multiples of the chip period, we have the chip-synchronous model; with τ_k continous random variable, in the range $[0, T]$, we have the most general chip-asynchronous system. We assume that all realization of temporal delays are known at the receivers and, without loss of generality, that $0 \leq \tau_1 \leq \cdots \leq \tau_K$.

A discrete model can be obtained by projecting the received signal onto the signal space spanned by the received signature, i.e. by condidering the outputs of a bank of filters matched to the signals $s_k(t - iT - \tau_k)$. The resulting vector model has the form

$$\mathbf{y} = \mathbf{R}\mathbf{A}\mathbf{b} + \sigma\mathbf{n}, \quad (2)$$

where $\mathbf{A}$ is an $MK \times MK$ diagonal matrix whose elements are the received amplitudes in a frame and the matrix $\mathbf{R}$ is an $MK \times MK$ block matrix containing all the partial cross-correlations between the signatures that depend on the relative delay between users, $\mathbf{b} = \left(\mathbf{b}^T[0]\ \mathbf{b}^T[1] \ldots \mathbf{b}^T[M-1]\right)^T$ is the MK vector of transmitted bits over the whole frame by all users, $\mathbf{n}$ is the MK-dimensional additive Gaussian noise [3]. Matrix $\mathbf{R}$ presents a well-defined structure that derives from the nature of the system and of the type of asynchronicity. We assume that all relative delays are known, because under this assumption the statistics expressed by the discrete-model (2) are sufficient [4].

The dimension of both $\mathbf{y}$ and $\mathbf{b}$ is MK and the model can be viewed as that of an equivalent synchronous system where the number of effective users equals MK. While impractical for any receiver implementation, because of the computational complexity that increases both with the number of users and the frame length (which may be very large), this model can be used to conduct an analysis of asynchronous systems by means of linear margins as done for synchronous systems in [1].

III. Asymptotic Multiuser Efficiency

A major drawback for any CDMA system is that when the multiuser interference becomes very strong there might exist a user that presents a probability of error that does not decrease to zero when the background noise is very small or reduced to zero. In other words the probability of error "floors" even if power increases. The asymptotic multiuser efficiency for a user k, usually denoted with η_k, is a paramenter defined by Verdú [3] that can reveal when these "floors" occur. In general η_k represents a measure of the effect of multiuser interference on the perfomance of user k and can assume values between 0 and 1. When $\eta_k = 0$ the multiuser interference dominates the (poor) performance of user k; when $\eta_k = 1$, user k has the same performance of the single user channel, therefore multiuser interference does not determine any degradation. We are concerned with the problem of "floors" in CDMA that occur when $\eta_k = 0$ when linear detectors are employed.

While Verdú provides a formal definition in [3], the asymptotic multiuser efficiency can also be viewed as the ratio of two energies: one is the energy required by user k to achieve the same performances of the single user channel when the background noise vanishes (asymptotic effective energy, AEE, see also [5]); the other is the energy spent by user k on the multiaccess channel, i.e.

$$\eta_k = \frac{E_k}{a_k^2}. \tag{3}$$

When the focus is restricted to linear receivers the AEE for user k can be expressed in terms of linear margins δ_k as defined by Romano *et al.* in [1]. The linear margin δ_k introduced for synchronous systems can be extended to the asynchronous system (2) and assume the same geometrical interpretation in terms of separability. The linear margin for the system model (2) is defined as

$$\delta_k = \frac{1}{\sqrt{\mathbf{c}_k^T \mathbf{R} \mathbf{c}_k}} \left(a_k \left|\mathbf{c}_k^T \mathbf{r}_k\right| - \sum_{j \neq k} a_j \left|\mathbf{c}_k^T \mathbf{r}_j\right| \right). \tag{4}$$

When δ_k is positive means that the hyperplane $\mathbf{c}_k$ geometrically separates the signal space into two sets of points: one with $b_k = +1$ and the other with $b_k = -1$ [1]. Similarly to the synchronous case, separability results in a necessary and sufficient condition to have a vanishing joint probability of error when the channel noise goes to zero. Romano *et al.* have also shown that when the system becomes overloaded then the multiuser interference is so high that any linear receiver is not able to correctly decode the received bits [1]. As for AEE, the linear margin δ_k depends in general on the signal space, the cross-correlations among the signatures and the amplitudes, and the receivers.

The relationship between E_k and δ_k can be easily derived by the expression of E_k for any linear receivers $\mathbf{c}_k^T$

$$E_k = \frac{1}{\mathbf{c}_k^T \mathbf{R} \mathbf{c}_k} \max{}^2 \left\{ 0, \left|\mathbf{c}_k^T \mathbf{r}_k\right| - \sum_{j \neq k} \frac{a_j}{a_k} \left|\mathbf{c}_k^T \mathbf{r}_j\right| \right\} \tag{5}$$

$$= \max{}^2 \left\{0, \delta_k\right\}. \tag{6}$$

Eq. (5) finds its justification in the expression of the probability of error for user k when linear detectors are employed. From eq. (6) we see that δ_k^2 represents the AEE when user k is separable and that $E_k > 0$ becomes a sufficient and necessary condition for separability. At least in the case of linear receivers, the linear margin represents a sort of distance from the separability condition to hold, i.e. it can tell how far from separability one user is.

The asymptotic multiuser efficiency for linear receivers becomes, under separability condition,

$$\eta_k = \left(\frac{\delta_k}{a_k}\right)^2. \tag{7}$$

Note that even though the efficiency may in general assume any value between 0 and 1 a more stringent range may be imposed by the signal space. Verdú in [6] shows that the optimum multiuser efficiency is given by

$$\eta_k^{\text{opt}} = \left(\frac{d_{k,\min}}{a_k}\right)^2 \tag{8}$$

where $d_{k,\min}$ is the minimum distance of the k-th user. From eq. (8) we can therefore derive an upper bound for δ_k

$$\delta_k \leq d_{k,\min}. \tag{9}$$

The asymptotic multiuser efficiency can be derived also from a matrix formulation of the linear margins. Consider the matrix

$$\mathbf{H} = \mathbf{D}^{-1/2} \mathbf{C}^T \mathbf{R} \mathbf{A} = \mathbf{D}^{-1/2} \hat{\mathbf{H}} \tag{10}$$

where $\mathbf{C}$ is a matrix whose columns are the linear receivers $\mathbf{c}_k^T$ and $\mathbf{D}$ is a diagonal matrix whose elements are the diagonal element of $\mathbf{C}^T\mathbf{R}\mathbf{C}$. Looking at each row of $\mathbf{H}$ we have that the AEE is

$$E_k = \left(d_{kk}^{-1}\right)\left(\left|\hat{h}_{kk}\right| - \sum_{j\neq i}\left|\hat{h}_{kj}\right|\right)^2. \tag{11}$$

Similarly, the asymptotic multiuser efficiency can be written as

$$\eta_k = a_{kk}^{-2} d_{kk}^{-1}\left(\left|\hat{h}_{kk}\right| - \sum_{j\neq i}\left|\hat{h}_{kj}\right|\right)^2. \tag{12}$$

AEE can also be obtained from the rows of matrix

$$\mathbf{E} = \mathbf{D}^{-1/2}\hat{\mathbf{H}} \tag{13}$$

and η_k from matrix

$$\mathbf{W} = \mathbf{A}^{-1}\mathbf{D}^{-1/2}\hat{\mathbf{H}}. \tag{14}$$

The study of asymptotic efficiency for any CDMA system (both synchronous and asynchronous) can be carried out by restricting the attention to the matrix $\hat{\mathbf{H}}$, since the sign of linear margins does not depend on the diagonal matrix $\mathbf{D}$. When the matrix $\hat{\mathbf{H}}$ is diagonally dominant all linear margins are positive. In [1] it has been shown that for overloaded synchronous systems the separability condition, that guarantees asymptotic efficiency, cannot be simultaneously achieved by all users and therefore the joint BER "floors". We want to extend our analysis to the overloaded asynchronous (both chip-synchronous and chip-asynchronous) with the aim of understanding whether the same limitations we have found for synchronous systems hold for asynchronous systems too.

A. Convetional detector

In asynchronous CDMA the conventional detector is made up of a bank of filters matched to the shifted signatures (under the assumption that delays are known) [3]. In general

$$\hat{\mathbf{H}} = \mathbf{R}\mathbf{A} \tag{15}$$

and the detector results asymptotically efficient when $\hat{\mathbf{H}}$ is diagonally dominant [1]. The diagonal dominance condition applied to the model (2) gives us the well-know open-eye condition for the asynchronous case

$$a_k > \sum_{j\neq k} a_j\left(|\rho_{jk}| + |\rho_{kj}|\right) \qquad k = 1,\ldots,K \tag{16}$$

Note that the diagonal dominance need to be checked for MK diagonal elements. But the specific structure of the matrix $\mathbf{R}$ (and $\mathbf{A}$) makes most of the condition found redundant since they repeat the same condition. This is not true however at the beginning and at the end of the frame, where we have a reduced multiuser interference because some of them do not transmit.

Note that differently from the synchronous case we cannot exclude that some combinations of received amplituded and partial cross-correlation make the conventional receiver asymptotically efficient even in overloaded scenario. This is because when the synchronous systems is overloaded the cross-correlation is equal to 1, while in asynchronous system, under the same overloaded condition, does not result $|\rho_{jk}| + |\rho_{kj}| = 1$, since the asynchronicity reduces the cross-correlations.

B. MMSE detector

The MMSE receiver is expressed by

$$\mathbf{C} = \left(\mathbf{R}\mathbf{A}^2\mathbf{R} + \sigma^2\mathbf{R}\right)^{-1}\mathbf{R}\mathbf{A} \tag{17}$$

and results, for $\sigma \to 0$,

$$\begin{aligned}\hat{\mathbf{H}} &= \mathbf{C}^T\mathbf{R}\mathbf{A}\\ &= \mathbf{A}\mathbf{R}\left(\mathbf{R}\mathbf{A}^2\mathbf{R} + \sigma^2\mathbf{R}\right)^{-1}\mathbf{R}\mathbf{A} = \mathbf{I},\end{aligned}$$

if the correlation matrix $\mathbf{R}$ is not singular, as for example in synchronous underloaded system. Other cases of interest in which $\mathbf{R}$ is not singular are when the system becomes overloaded in asynchronous systems. In other words as long as the matrix $\mathbf{R}$ is not singular the MMSE detector is always asymptotically efficient. The important result here is that for overloaded asynchronous CDMA systems we find a linear receiver that is always asymptotically efficient, suggesting that for such systems is better to be asynchronous than synchronous.

We compute now the AEE and η for the MMSE, deriving a new expression for the asymptotic efficiency which puts on evidence the role of the sole correlation matrix $\mathbf{R}$ in the asymptotic efficiency for asynchronous CDMA. Matrices $\mathbf{E}$ and $\mathbf{W}$, that we have introduced before, becomes diagonal as they can be written as

$$\mathbf{E} = \mathbf{D}^{-1/2} \tag{18}$$

and

$$\mathbf{W} = \mathbf{A}^{-1}\mathbf{D}^{-1/2}. \tag{19}$$

Since AEE for user k is the corresponding diagonal element of matrix $\mathbf{E}$ and $\mathbf{C}^T\mathbf{R}\mathbf{C}$ has the same diagonal of $\mathbf{E}$, we can express the AEE starting from $\left(\mathbf{C}^T\mathbf{R}\mathbf{C}\right)^{-1}$ taking the limit for $\sigma \to 0$. Hence we have that

$$\begin{aligned}\lim_{\sigma\to 0}\mathbf{C}^T\mathbf{R}\mathbf{C} &= \mathbf{A}\mathbf{R}\left(\mathbf{R}\mathbf{A}^2\mathbf{R} + \sigma^2\mathbf{R}\right)^{-1}\mathbf{R}\\ &\quad\left(\mathbf{R}\mathbf{A}^2\mathbf{R} + \sigma^2\mathbf{R}\right)^{-1}\mathbf{R}\mathbf{A} = \left(\mathbf{A}\mathbf{R}\mathbf{A}\right)^{-1}\end{aligned} \tag{20}$$

from which we obtain the known result [5]

$$E_k = \frac{1}{\left(\left(\mathbf{A}\mathbf{R}\mathbf{A}\right)^{-1}\right)_{kk}} \tag{21}$$

Since the asymptotic efficiency is given by the ratio of E_k and a_k^2 we have

$$\eta_k = \frac{1}{a_k^2\left(\left(\mathbf{A}\mathbf{R}\mathbf{A}\right)^{-1}\right)_{kk}} \tag{22}$$

According to the definition of adjoint matrix we have

$$\left(\left(\mathbf{A}\mathbf{R}\mathbf{A}\right)^{-1}\right)_{kk} = \frac{\det\left(\mathbf{A}_k\mathbf{R}_k\mathbf{A}_k\right)}{\det\left(\mathbf{A}\mathbf{R}\mathbf{A}\right)} \tag{23}$$

where $\mathbf{A}_k$ and $\mathbf{R}_k$ denote the matrices $\mathbf{A}$ and $\mathbf{R}$ without the k-th row and column, i.e. when user k is eliminated from the channel. Eq. (22) can be written as

$$\eta_k = \frac{\det(\mathbf{ARA})}{a_k^2 \det(\mathbf{A}_k \mathbf{R}_k \mathbf{A}_k)} \tag{24}$$

$$= \frac{\left(\prod_{i=1}^{K} a_i\right)^2 \det(\mathbf{R})}{a_k^2 \left(\prod_{i=1, i\neq k}^{K} a_k\right)^2 \det(\mathbf{R}_k)} \tag{25}$$

$$= \frac{\det(\mathbf{R})}{\det(\mathbf{R}_k)}. \tag{26}$$

Eq. (26) is a genereal results that remains valid as long as the matrix $\mathbf{R}$ is not singular. Since in underloaded synchronous systems $\mathbf{R}$ is not singular we find the well-known result that for such systems the MMSE receiver (as well as the decorrelator, which has the same asymptotic performances of MMSE) is always asymptotically efficient.

Application of the Hadamard-Fischer's inequality gives us an upper bound for eq. (26)

$$\eta_k \leq \min\left\{\min_{j\neq k}\left(1-\rho_{k,j}^2\right), \min_{j\neq k}\left(1-\rho_{j,k}^2\right)\right\} \qquad j,k = 1,\ldots,K \tag{27}$$

where $\rho_{k,j}$ and $\rho_{j,k}$ are the partial cross-correlations between users j and k. Note that the bound (27) is zero when any of the partial cross-correlation is equal to 1, having as consequence that the MMSE receiver cannot be asymptotically efficient. While in synchronous systems this happens when the system is overloaded, in chip-asynchronous systems this happens depending on the specific realization of delays, because partial cross-correlations depend on the relative delay between users. Since partial cross-correlations are equal to 1 for a countable set of events, the probability of these events is zero as long as the delay is a continous random variable. Instead, in chip-synchronous systems, because of the discrete nature of the temporal delays, the probability of obtaining partial cross-correlations equal to 1 is not null.

IV. Simulations

We show in this section some results from simulation of synchronous and chip-asynchronous CDMA systems. We have first considered a CDMA system with processing gain $N = 2$ and randomly generated signatures (equal for all the set of simulations) and equal power. The number of user is $K = 4$ and the frame length is $M = 8$. The receiver employed is an MMSE detector. Fig. 1 show different plots of the probability of error for a synchronous system and for the corresponding chip-asynchronous system obtained with different realization of relative delays.

Results show that in overloaded system, i.e. with a number of users greater than the processing gain, the chip-asynchronous system can be asymptotically efficient, avoiding the "floor" that the synchronous system always presents. Different realization of the relative delays introduce an additional degree of diversity whose effect in terms of joint BER changes

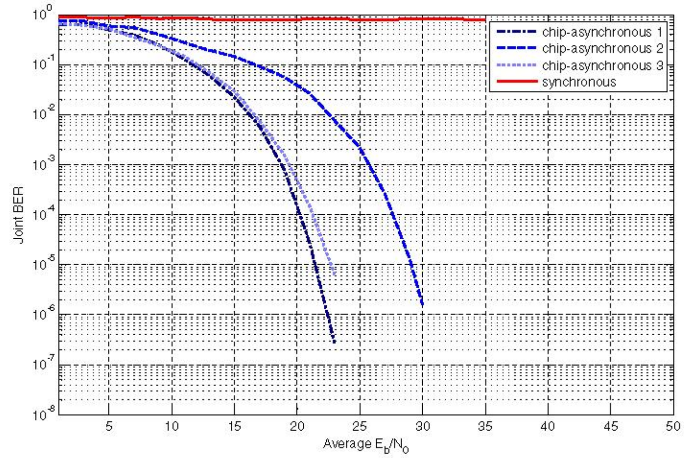

Figure 1. Comparison of synchronous and chip-asynchronous overloaded systems, with MMSE detector. Chip-asynchronous curves are obtained for different realization of delays: $\tau_1 = \{0.20, 0.68, 0.93, 1.94\}$, $\tau_2 = \{0.05, 0.50, 1.80, 1.87\}$ and $\tau_3 = \{0.02, 0.52, 1.74, 1.83\}$

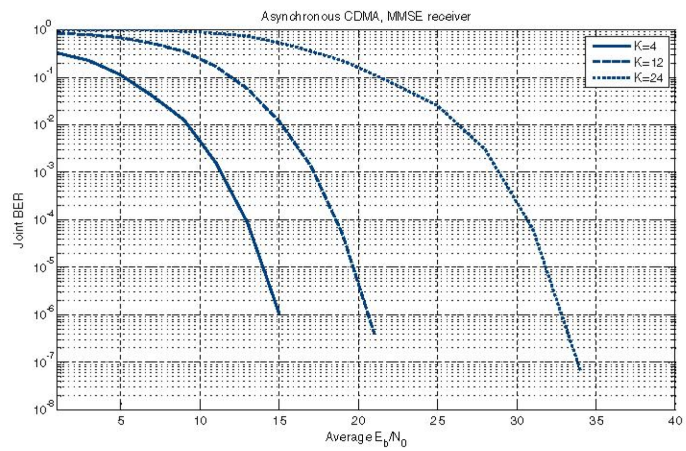

Figure 2. Asynchronous CDMA with MMSE receiver for increasing number of users.

somehow. However, as far as we are concerned with the asymptotic performances it is clear that in overloaded system is better to be asynchronous than synchronous, at least with MMSE receiver.

We have also considered an asynchronous CDMA system, with the MMSE receiver, with an increasing overloading factor K/N, to make evident that as long as the system becomes more overloaded performances degrade but we do not have a "floor". We report serveral plots of the joint probability of error versus the average $\mathcal{E}_b/N_0$, for a system with fixed processing gain $N = 4$, frame duration $M = 4$ bits, and randomly generated signatures which remain fixed for the whole sets of simulations. Results are shown for a number of users $K = 4, 12, 24$ in Fig. 2.

Results show that an increase of the number of users does not determine a "floor", even if we pass from a fully loaded ($N = 4, K = 4$) system to a heavy overloaded system ($N = 4, K = 24$), at the expenses of a much higher SNR that is needed to mantain the same performances. As an example, for a probability of error equals to 10^{-6}, for $K/N = 3$ we need an increase of about 3dB in SNR to obtain the same preformance of the fully loaded system.

V. Conclusions

Linear detectors present strong limitations in synchronous CDMA systems when the number of users exceeds the processing gain. In such systems there exists at least one user that has a probability of error that does not decrease as the background noise vanishes, because the multiuser interference is so high that linear detectors are not able to cancel it out for all. When the system becomes chip-asynchronous linear detectors do not present such strong limitations. At least for the conventional and MMSE detectors, which we have considered in this paper, we do not have always the emergence of a "floor". Further we have shown that the MMSE detector is asymptotically efficient with probability 1 even in overloaded scenarios.

Therefore an overloaded chip-asynchronous system may perform better than the correspondent synchronous system, since a system, with more users than signatures, that becomes asynchronous, looses its overloaded nature because each user's delay increases the system diversity.

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